The Billingsley Lemma. Dimension of self-affine sets

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Now, let us generalize Mass Distribution Principle.

For $x \in \mathbb{R}^d$, let $Q_n(x)$ be b^{-n} b-adic cube containing x, as before.

Lemma (Billingsley)

Let K be a Borel set in $\mathbb{R}^d$, μ - finite Borel measure on $\mathbb{R}^d$ and $\mu(K) > 0$. If for some $\beta \geq \alpha \geq 0$

$$\alpha \leq \lim_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n \log b} \leq \beta \quad \forall x \in K, \text{ then } \alpha \leq \operatorname{Hdim} K \leq \beta.$$

Pf. Take $\alpha_1 < \alpha$. By the left inequality,

$\lim_{n\to\infty} b^{\alpha_1 n} \mu(Q_n(x)) = 0 \ \forall x \in K$. By Singularity lemma applied to $h(x) = x^{\alpha_1}$, $m_{\alpha_1}(K) = \infty$, so $\operatorname{Hdim} K \geq \alpha, \forall \alpha_1 < \alpha$.

In the other direction, for $\beta_1 > \beta$, we have

$\overline{\lim}\, b^{\beta_1 n} \mu(Q_n(x)) = \infty$ for all $x \in K$.

Fix $\varepsilon > 0$, and for any $x \in K$ let $n(x)$ be the smallest n such that $b^{\beta_1 n} \mu(Q_n(x)) > (\sqrt{d})^{\beta_1}$ and $\sqrt{d}\, b^{-n} \leq \varepsilon$.

$(Q_{n(x)})_{x\in K}$ is a cover of K. Select a non-intersecting subcover (Q_k). Then $\operatorname{diam} Q_k < \varepsilon$ (by the choice of $n(x)$),

$\sum (\operatorname{diam} Q_k)^{\beta_1} \leq \sum \mu(Q_k) \leq \mu(\mathbb{R}^d)$. Thus

$m_{\beta_1, \varepsilon}(K) \leq \mu(\mathbb{R}^d)$. Take $\varepsilon \to 0$, to get that $m_{\beta_1}(K) < \infty$ ∎

An application: dimension of lacunary set.

Let us show that

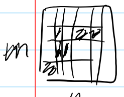
$\operatorname{Hdim} A_S = \underline{d}(S)$.

Let us define μ to be – measure giving equal weight $2^{-\#(S \cap \{1,..n\})}$ to each of the n-th generation intervals of C_n. For $x \in C$,

$$\underline{\lim_{n\to\infty}} \frac{\log \mu(Q_n(x))}{\log 2^{-n}} = \underline{d}(S).$$ So, by Billingsley, $\operatorname{Hdim} A_S = \underline{d}(S)$ ∎

Another application: self-similar sets.

Let D be an $n \times m$ pattern on the square:



Formally, $D \subset \{0,..,n-1\} \times \{0,..,m-1\}$

Repeat the pattern on every non-removed rectangle.

$K(D) := \{ \sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}) : (a_k, b_k) \in D \}$.

Most famous: McMullen set, corresponding to $D = \{(0,0), (1,1), (2,0)\}$ in 3×2 rectangle.

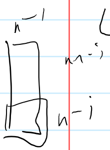
Lemma (Easy). Assume $n \geq m$ and that every row contains a chosen rectangle (i.e. $\forall j\ \exists i : (i,j) \in D$).

Then $\operatorname{Mdim} K(D) = 1 + \log_n \frac{\#(D)}{m}$.

Exercise: What happens when not every row contains a chosen rectangle?

Proof. Let $r := \#(D)$ and ~~let~~ us look at the j-th stage of the

Proof. Let $r := \#(D)$ and ~~let us~~ look at the j-th stage of the construction. It consists of r^j elongated rectangles of width n^{-j} and height m^{-j}.



Let $k = \left[\frac{\log n}{\log m} j\right]$. Then we need $\frac{m^{-j}}{n^{-j}} = m^{k-j}$ squares of size m^{-k} to cover each such rectangle, and we cannot use less than m^{k-j-1} (<u>because every row has a rectangle in it</u>, thus the height of $K(D) \cap$ any given rectangle is still m^{-j}!).

So $N(K(D), m^{-k}) \approx r^j m^{k-j}$,

$$\mathrm{Mdim}\, K(D) = \lim_{k\to\infty} \frac{\log r^j + \log m^{k-j}}{k \log m} = \frac{\log r}{\log n} + \frac{\log n}{\log n} - \frac{\log m}{\log n} = 1 + \log_n \frac{r}{m}$$

So, for McMullen set, $\mathrm{Hdim}\, K(D) = 1 + \log_3 \frac{3}{2}$.

But

Thm. In the assumptions of the Lemma, $\mathrm{Hdim}(K(D)) = \log_m \left(\sum r(j)^{\log_n m}\right)$ where $r(j) = \#\{i : (i,j) \in D\}$ - number of rectangles lying in a given row.

It is easy to see that $\mathrm{Hdim}(K(D)) \le \mathrm{Mdim}(K(D))$, with equality reached iff $r(j)$ does not depend on j.

For McMullen set, $\dim K(D) = \log_2\left(1 + 2^{\log_3 2}\right)$

Pf Instead of b-adic squares, we use more appropriate approximate squares

$\alpha := \frac{\log m}{\log n}$

$$Q_k(x,y) = \left\{ (x', y') : \begin{array}{ll} x' = \sum x_i' n^{-i} & x_i' = x_i,\ j \le \lfloor \alpha k \rfloor \\ y' = \sum y_i' m^{-i} & y_i' = y_i,\ j \le k \end{array} \right\}.$$

We expand x in base n! y in base m!

It is almost a square, since $m^{-k} \le n^{-\lfloor \alpha k \rfloor} \le n m^{-k}$, and $\mathrm{diam}\, Q_k \cong m^{-k}$ (up to a factor of $\sqrt{2}\, n$).

Given a probability vector $\vec{p}$ with coordinates $p(d)$, $d \in D$, define the probability measure $\mu_{\vec{p}}$ on $K(D)$ by the usual iterative process. Assume also that $\tilde{p}(d_1) = \tilde{p}(d_2)$ if d_1, d_2 are in the same row (i.e. $y_{d_1} = y_{d_2}$).

Thus if $d = (x, y)$, $x \in \{0, \dots, n-1\}$, $y \in \{0, \dots, m-1\}$, $\tilde{p}(d) = p(j)$, $\sum r(j) p(j) = 1$.

$$\mu_{\vec{p}}\, Q_k(x,y) = \prod_{j=1}^{k} p(y_j) \prod_{j=\lfloor \alpha k\rfloor+1}^{k} r(y_j)$$

(since $Q_k(x,y)$ contains exactly $\prod_{j=\lfloor \alpha k \rfloor+1}^{k} r(y_j)$ rectangles of level k, each of the same measure).

Thus $\log(\mu_{\vec{p}}(Q_k(x,y))) = \sum_{j=1}^{k} \log p(y_j) + \sum_{j=\lfloor \alpha k\rfloor+1}^{k} \log r(y_j)$

Then, since (y_i) are i.i.d. wrt $\mu_{\vec{p}}$, by SLLN, (or by ergodic Thm, discussed)

$$\lim_{k\to\infty} \frac{1}{k} \log(\mu_{\vec{p}}(Q_k(x,y))) = \sum_j r(j) p(j) \log p(j) + (1-\alpha) \sum_j r(j) p(j) \log r(j) \quad \mu_{\vec{p}}\text{-a.e.}$$

and, by a version of Billingsley's Lemma for approximate squares,

$$\mathrm{Hdim}\, K(D) \ge \frac{1}{\log m} \sum_j r(j) p(j) \left(\log \frac{1}{p(j)} + \log \left(r(j)^{\alpha - 1}\right) \right).$$

To maximize it pick

$\mathrm{Hdim}\, K(D) \geq \frac{1}{\log m} \sum_{j} p(j) \left(\log \frac{1}{p(j)} + \log (r(j)^{\alpha-1}) \right).$

To maximize it pick

$$p(j) = \frac{1}{Z} r(j)^{\alpha-1}, \quad Z = \sum_{d \in D} r(d)^{\alpha-1} = \sum_{j=0}^{m-1} r(j)^{\alpha}.$$

Then $\mathrm{Hdim}\, K(D) \geq \log_m Z$, and, thus we obtain the right lower bound for $\mathrm{Hdim}\, K(D)$.

For the upper bound, if

$$S_k(x,y) := \sum_{j=1}^{k} \log r(y_j).$$

$\frac{1}{k} S_k(x,y)$ is uniformly bounded,

$$\log \mu(Q_k(x,y)) = \sum_{j=1}^{k} \log \frac{1}{Z} r(y_j)^{\alpha-1} + \left(\sum_{j=1}^{k} \log r(y_j) - \sum_{j=1}^{\lfloor \alpha k \rfloor} \log r(y_j) \right) =$$

$$= -\sum_{j=1}^{k} \log Z + (\alpha-1) S_k(x,y) + S_k(x,y) - S_{\lfloor \alpha k \rfloor}(x,y), \text{ or}$$

$$\log \mu(Q_k(x,y)) + k \log Z = \alpha S_k(x,y) - S_{\lfloor \alpha k \rfloor}(x,y).$$

So:

$$\frac{1}{\alpha k} \log \mu(Q_k(x,y)) + \frac{1}{\alpha} \log Z = \frac{S_k(x,y)}{k} - \frac{S_{\lfloor \alpha k \rfloor}(x,y)}{\alpha k}.$$

Now do the telescoping series to get that, since

$$\sum \frac{S_{\alpha^{-j}}(x,y)}{\alpha^{-j}} - \frac{S_{\alpha^{-j+1}}(x,y)}{\alpha^{-j+1}} \text{ converges,}$$

$$\overline{\lim} \left(\frac{S_k(x,y)}{k} - \frac{S_{\lfloor \alpha k \rfloor}(x,y)}{\alpha k} \right) \geq 0. \Rightarrow$$

$$\overline{\lim} \left(\log \mu(Q_k(x,y)) + k \log Z \right) \geq 0 \Rightarrow$$

$$\underline{\lim} \frac{\log \mu(Q_k(x,y))}{-k} \leq \log Z. \text{ So, since}$$

$\mathrm{diam}\, Q_k(x,y) \asymp m^{-k}$, $\mathrm{Hdim}(K(D)) \leq \log_m Z$

(we use again an approximate square version of Billingsley's Lemma)